Stem and topological entropy on Cayley trees

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Motivations: statistical physics

- $\mathcal{A}$: symbol sets, alphabets
- **Boltzmann distribution**: $X \subseteq \mathcal{A}^N$ **configuration space**, $E : X \to \mathbb{R}$, **energy function**

$$p_\beta(x) = \frac{1}{Z(\beta)} e^{-\beta E(x)}, \ Z(\beta) = \sum_{x \in X} e^{-\beta E(x)}.$$

- Temperature $T = \frac{1}{\beta}$ (β: **inverse temperature**)
- $Z(\beta)$: **partition function**
- $\beta \to 0$ (high-temperature limit), the flat probability distribution

$$\lim_{\beta \to 0} p_\beta(x) = \frac{1}{|\mathcal{A}|}$$

- $\beta \to \infty$ (low-temperature limit), find the ground state, i.e., $E(x) \geq E(x_0)\ \forall x$

Free energy and free entropy

- **Free energy**

$$F(\beta) = \frac{-1}{\beta} \log Z(\beta),\ \Phi(\beta) = -\beta F(\beta) = \log Z(\beta)$$

- **Internal energy**

$$U(\beta) = \frac{\partial}{\partial \beta}(\beta F(\beta))$$

- **Canonical entropy**

$$S(\beta) = \beta^2 \frac{\partial F(\beta)}{\partial \beta}$$

Variational principle and entropy

- Variational principle and some known facts:

$$\begin{aligned}F(\beta) &= U(\beta) - \frac{1}{\beta}S(\beta) = -\frac{1}{\beta}\Phi(\beta)\\ U(\beta) &= \langle E(x)\rangle \\ S(\beta) &= -\sum_x p_\beta(x)\log p_\beta(x)\\ -\frac{\partial^2}{\partial\beta^2}(\beta F(\beta)) &= \left\langle E(x)^2\right\rangle - \langle E(x)\rangle^2 .\end{aligned}$$

Free energy density and free entropy density

- **Free energy density, pressure function:** For $N \to \infty$

$$f(\beta) = \lim_{N\to\infty} \frac{F_N(\beta)}{N},$$

- **Free entropy density, metric entropy:**

$$s(\beta) = \lim_{N\to\infty} \frac{S_N(\beta)}{N}.$$

Ising potential and its Boltzmann distribution

- **Ising potential:**

$$E(x) = -\sum_{(i,j)\in E} x_i x_j - \sum_{i\in G} x_i,$$

where E : nearest neighbors couples (i,j) such that $i,j \in G$

- **Boltzmann distribution**:

$$\begin{aligned}\mu(x) &= \frac{1}{Z(\beta)} \exp\{\beta(\sum_{(i,j)\in E} x_i x_j + \sum_{i\in G} x_i)\} \\ &= \frac{1}{Z(\beta)} \exp(-\beta E(x)),\end{aligned}$$

where $Z(\beta)$ is the partition function, i.e., the normalization such that $\sum_x \mu(x) = 1$.

G is a countable graph

- $\mathcal{A}$: symbol set, G is a countable graph and $X \subseteq \mathcal{A}^G$ is a configuration space.
- **Free energy density**: Given $E : X \to \mathbb{R}$ is a potential function and $\beta \in \mathbb{R}$. Let
$$\Lambda_n = \{g \in G : |g| \leq n\}.$$
The free energy density is defined as
$$f(\beta) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \log Z_n(\beta),$$
where $Z_n(\beta)$ is the partition function on Λ_n; that is
$$Z_n(\beta) = \sum_{x\in X|_{\Lambda_n}} e^{-\beta E(x)}.$$

Main problems

Problem

Let $X \subseteq \mathcal{A}^G$, and $E : X \to \mathbb{R}$ be a potential function, when the limit

$$f_E(\beta) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \log Z_n(\beta)$$

exists?

- If $\beta = 0$, then

$$f_E(0) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \log \sum_{x\in X|_{\Lambda_n}} \exp(-0E(x)) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \log \sum_{x\in X|_{\Lambda_n}} 1.$$

Problem (Our main works)

When $f_E(0)$ exists?

Phase transition

- **Phase transition:**
  - The free energy density is continuous, but its derivative w.r.t β is discontinuous at β_c. This singularity is named a **first order phase transition**.
  - The free energy and its first derivative are continuous, but the second derivative is discontinuous at β_c. This is called a **second order phase transition**.
- Thus, the existence of the limit of $f_E(\beta)$ is the first and importnat step.
- Next step is to compute the explicite formula of $f_E(\beta)$.
- For G is an arbitrary countable group. Prove the existence of $f_E(\beta)$ and find its formula are an extremely difficult problem.
- (information theory) If X is the Markov chain, the value of $f_E(0)$ is also known as the **Shannon-McMillan-Breiman theorem**, **entropy ergodic theorem** or **asymptotic equipartition property** (**AEP**)

Previous works

- Entropy Ergodic Theorem:
  - $\mathbb{Z}^1$: Shannon, 1948, in probability; McMillan, 1953, L^1; Breiman, 1957, a.e.; Chung, 1961, countable alphabets; Yang, 1998, nonhomogeneous; Wang-Yang, 2016, nonhomogeneous; Yang-Liu, 2004, mth order nonhomogeneous.
  - Tree: Berger-Ye, 1990; Yang 2003, Berger, 1996; Berger 1998; Yang-Liu, 2000
- Ising model on $\mathbb{Z}^d$ and trees: Preston, 1974; Gerigii, 2011; Spitzer,75; Pemantle, 1992; Kemeny-Snell, 1976; Lyons, 2000; Dembo, 2010; Dembo et al, 2010, Georgii, 2011.
- Tree dynamics: Benjamini-Peres, 1974; Mossel, 1998; Mezard-Montanari, 2005; Mossel-Peres, 2003; Aubrun-Beal, 2012, 2013, 2014; Petersen-Salama, 2017, 2018; Ban-Chang, 2017a, 2017b, 2019.
- Nice references
  - C. J. Preston: Gibbs states on countable sets, 1974.
  - H.O. Georgii: Gibbs measures and phase transitions, 2011.

Preliminaries

- G : finitely generated semigroup, $S_k = \{s_1, s_2, \ldots, s_k\} \subset G$
- $K \in \mathcal{M}_k(\{0,1\})$, $G = \langle S_k | R \rangle$, and $R = \{s_i s_j : K(s_i, s_j) = 0\}$
- $\mathcal{T}$: the Cayley graph of G, i.e., the vertex set is G and the edge set is $E = \{(g, gs) : g \in G,\ s \in S_k\}$.
- **Labeled tree**: Let $\mathcal{A}$ be a finite alphabet. A leveled tree is a function $t : G \to \mathcal{A}$ for which $t_g = t(g)$ is the label attached to $g \in G$.
- **Pattern**: A pattern is a function $u : H \to \mathcal{A}$ for some finite set $H \subset G$, where $s(u) := H$ is the **support** of u.
- A pattern is **accepted** by $t \in \mathcal{A}^G$ if $\exists g \in G$ such that $t|_{gs(u)} = u$. Otherwise, t **rejects** u.
- $X \subseteq \mathcal{A}^G$ is a **tree shift** if $\exists \mathcal{F}$ (**forbidden sets**) such that t rejects $\forall u \in \mathcal{F}$ and $\forall x \in X$. Write $X = X_{\mathcal{F}}$.

Preliminaries

- A tree shift X is a **tree shift of finite type** (TSFT) if $X = X_{\mathcal{F}}$ for some finite forbidden set $\mathcal{F}$.
- Let $\mathbf{A} = (A_1, \ldots, A_k)$ be a k-tuple of binary matrixes indexed by $\mathcal{A}$, a **Markov tree shift** $X_A \subseteq \mathcal{A}^G$ is defined

$$X_{\mathbf{A}} = \{t \in \mathcal{A}^G : A_i(t_g, t_{gs_i}) = 1, \forall g \in G, |gs_i| = |g| + 1\}$$

- For $g \in G$ and $n \geq 0$,define the n-**ball centered at** g (the n-**semiball centered at** g)

$$\begin{aligned} \Delta_n^{(g)} \quad &: \quad = \{gh : h \in G, |h| \leq n\}. \\ \overline{\Delta}_n^{(g)} \quad &: \quad = \{gh : h \in G, |h| \leq n, |gh| = |g| + |h|\}. \end{aligned}$$

- Define

$$\overline{\Delta}_n^{(s_i)+} := \{s_i h : h \in G, |h| \leq n, |s_i h| = 1 + |h|\} \cup \{1_G\}.$$

Preliminaries

- Suppose $g \in G$, $a \in \mathcal{A}$, define

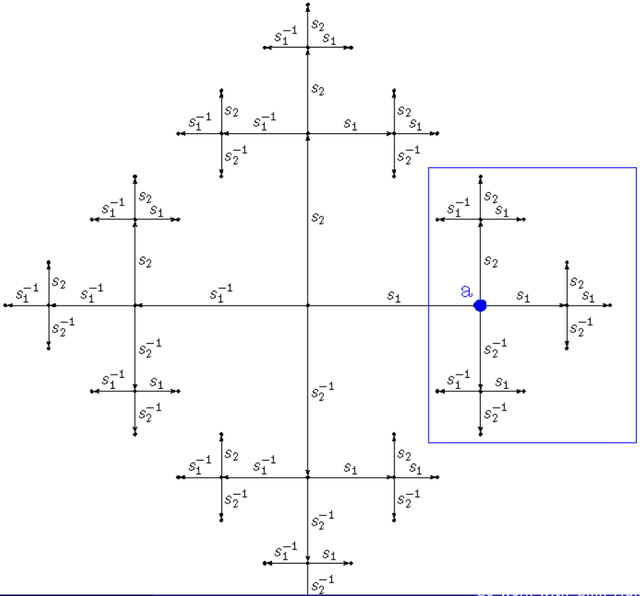
$$\begin{aligned} B_n^{(g)} &: = \{u \in \mathcal{A}^{\Delta_n^{(g)}} : u \text{ is accepted by some } t \in X\}; \\ B_{n;a}^{(g)} &: = \{u \in B_n^{(g)} : u_g = a\}; \\ B_n &: = B_n^{(1_G)}, B_{n;a} := B_{n;a}^{(1_G)}. \end{aligned}$$

Preliminaries

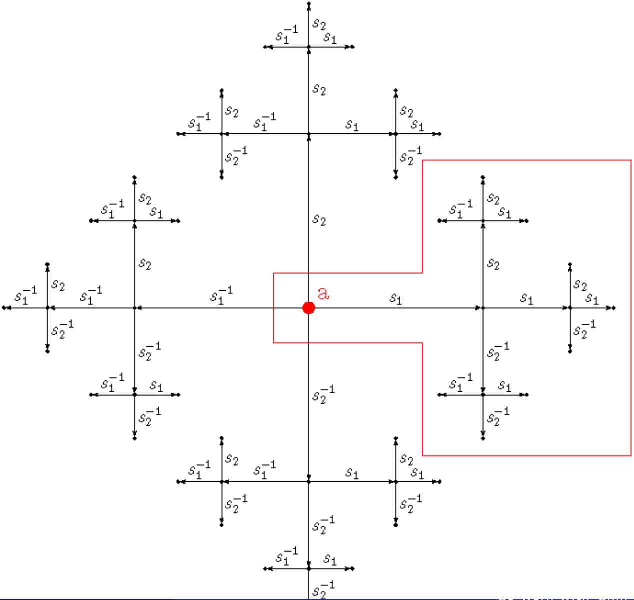
- Define

$$
\begin{aligned}
C_n^{(g)} \quad &: \quad = \{u \in \mathcal{A}^{\overline{\Delta}_n^{(g)}} : u \text{ is accepted by some } t \in X\}; \\
C_n^{(s_i)+} \quad &: \quad = \{u \in \mathcal{A}^{\overline{\Delta}_n^{(s_i)+}} : u \text{ is accepted by some } t \in X\}; \\
C_{n;a}^{(g)} \quad &: \quad = \{u \in C_n^{(g)} : u_g = a\}; \\
C_{n;a}^{(s_i)+} \quad &: \quad = \{u \in C_n^{(s_i)+} : u_g = a\}; \\
p_n^{(g)} \quad &: \quad = \left| C_n^{(g)} \right|, p_{n,a}^{(g)} = \left| C_{n;a}^{(g)} \right|; \\
q_n^{(s_i)} \quad &: \quad = \left| C_n^{(s_i)+} \right|, q_{n;a}^{(s_i)} := \left| C_{n;a}^{(s_i)+} \right|.
\end{aligned}
$$

- $C_{2;a}^{(S_1)}$



- $C^{(s_1)+}_{3;a}$



Stem entropy and topological entropy

- The **i-th stem entropy** of X :

$$h^{(s_i)} = h^{(s_i)}(X) := \limsup_{n\to\infty} \frac{\log p_n^{(s_i)}}{\left|\overline{\Delta}_n^{(s_i)}\right|}. \tag{1}$$

- If $h^{(s_i)} = h^{(s_j)}\ \forall i, j$, we call **stem entropy** and denoted by $h^{(s)}$.
- The **topological entropy** of X is defined as

$$h = h(X) := \lim_{n\to\infty} \frac{\log |B_n|}{|\Delta_n|},$$

provided the limit exists.

Existence of the stem entropy

Theorem (Existence of stem entropy)

Suppose that $G = \langle S_k | K \rangle$ is finitely generated semigroup, and $X \subset \mathcal{A}^G$ is a shift space on G. If K is primitive, then the stem entropy of X exists. In other words, for $1 \leq i, j \leq k$,

$$\limsup_{m \to \infty} \frac{\log p_m^{(s_i)}}{\left| \overline{\Delta}_m^{(s_i)} \right|} = \limsup_{m \to \infty} \frac{\log p_m^{(s_j)}}{\left| \overline{\Delta}_m^{(s_j)} \right|}.$$

Existence of the i-th stem entropy

Theorem (Existence of the i-th stem entropy)

Suppose that $G = \langle S_k | K \rangle$ is finitely generated semigroup, and $X \subset \mathcal{A}^G$ is a shift space on G. If K is primitive, then the limit of the ith-stem entropy of X (1) exists, and

$$\lim_{n\to\infty} \frac{\log p_n^{(s_i)}}{\left|\overline{\Delta}_n^{(s_i)}\right|} = \inf_{n\geq 0} \max_{1\leq j\leq k} \frac{\log p_n^{(s_i)}}{\left|\overline{\Delta}_n^{(s_i)}\right|} \text{ for } 1 \leq i \leq k.$$

Existence of topological entropy: full row

Theorem (Existence of topological entropy: full row)

Suppose $K \in \{0,1\}^{k\times k}$ satisfies $\sum_{j=1}^{k} K(s_i, s_j) = k$ for some $s_i \in S_k$, and X is a Markov tree shift. Then the topological entropy of X exists and

$$h = \lim_{n\to\infty} \frac{\log |B_n|}{|\Delta_n|} = h^{(s)}.$$

Application to shifts on golden-mean tree

Corollary (B-Chang-Huang, 2020, JAC)

Suppose G is generated by S_2 with $K = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, and X is a Markov tree shift. Then the topological entropy of X exists and can be calculated via a system of nonlinear recurrence equations.

Existence of topological entropy: equal row

Theorem (Existence of topological entropy: equal row)

Suppose $m = \sum_{j=1}^{k} K(s_i, s_j) = \sum_{j=1}^{k} K(s_{i'}, s_j)$ *for every* $1 \leq i, i' \leq k$ *and* $X = X_{\mathbf{A}}$ *is a hom Markov tree shift. Then the topological entropy exists and*

$$\lim_{n\to\infty} \frac{\log |B_n|}{|\Delta_n|} = h^{(s)}.$$

Example

Bethe lattice, for which the matrices K's have each diagonal entry 0 and each non-diagonal entry 1.

Application to shifts on free groups

Corollary (Free groups: hom shifts)

Let $G = F_k$ be a free group of rank k. That is, $G = \langle S_{2k} | K \rangle$ with $K(s_i, s_j) = 0$ if and only if $|i - j| = k$. Suppose $X = X_{\mathbf{A},\mathbf{A}^t}$ is a Markov shift space over F_k with $A_1 = A_2 = \cdots = A_k = A$ indexed by a finite alphabet $\mathcal{A}$. Then the limit $\lim_{n\to\infty} \frac{\log|B_n|}{|\Delta_n|}$ exists and equals $h^{(s)}$.

Corollary (Free groups: non-hom shifts)

Suppose $\mathcal{A}$ is a finite alphabet with $|\mathcal{A}| \leq 2k - 1$. Let $X_{\mathbf{A},\mathbf{A}^t}$ be a Markov shift over F_k with $\mathbf{A} = (A_1, A_2, \ldots, A_k)$. Then the topological entropy of X exists and equals $h^{(s)}$.

Generalization of mixing property

- Let $G = \langle S_k | K \rangle$ be a finitely generated semigroup. Suppose $X = X_{\mathbf{A}} \subseteq \mathcal{A}^G$ is a Markov tree shift on G. A **graph representation** of X is a directed graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ with vertex set $\mathbf{V} = \mathcal{A} \times S_k$ and with the edge set $\mathbf{E} = \{((a, s_i), (a, s_j)) \in \mathbf{V} \times \mathbf{V} : K(s_i, s_j) = 1, A_j(a, b) = 1\}$.
    - $\mathbf{G}$ is **strongly connected** if for every (a, s_i), $(a, s_j) \in \mathbf{V}$ there is a walk of length N from (a, s_i) to (b, s_j) in $\mathbf{G}$ (denoted $(a, s_i) \overset{N}{\twoheadrightarrow} (b, s_j)$) for some N depending (a, s_i) and (b, s_j).
    - A vertex $(a, s_i) \in \mathbf{V}$ is called a **pivot** if there exists $s_j \in S_k$ and an integer $N \in \mathbb{N}$ such that every $(b, s_j) \in \mathbf{V}$ admits a walk $(a, s_i) \overset{N}{\twoheadrightarrow} (b, s_j)$.

Equivalences of mixing properties

Theorem (Equivalence for hom shifts)

Suppose that $X_{\mathbf{A}} \subseteq \mathcal{A}^G$ is a hom Markov tree shift, and $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ is a graph representation of $X_{\mathbf{A}}$. Then

1. $\mathbf{G}$ *is strongly connected if and only if A is irreducible.*
2. $\mathbf{G}$ *is strongly connected and contains a pivot if and only if A is primitive.*

Mixing and existence of topological entropy

Theorem (Mixing and existence of the topological entropy)

Let $X_{\mathbf{A}} \subseteq \mathcal{A}^G$ be a Markov tree shift on G. Suppose $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ is a graph representation of $X_{\mathbf{A}}$. Then the topological entropy $h = \lim_{n\to\infty} \frac{\log|B_n|}{|\Delta_n|}$ exists and $h = h^{(s)}$ provided $\mathbf{G}$ admits a pivot and is strongly connected.

Corollary (Petersen-Salama, 2019, TCS)

If $X_{\mathbf{A}}$ is a hom Markov tree shift, then the topological entropy h exists and equals $h^{(s)}$ if A is primitive.